Features of the quasienergy spectrum of the hydrogen atom in a microwave field

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The nearest neighbour level spacing of the quasienergies of a hydrogen atom in a microwave field is studied. There is evidence of level repulsion among states corresponding to classically chaotic regions. In the regions of classical phase space where the motion is approximately regular, the corresponding quasienergy spectrum can be understood in terms of approximate dynamical constants. In particular, grouping the high lying quasienergy states into photons yields level spacing plots that are close to $\delta$ functions; there is little fluctuation about the average value predicted by the approximate constant. The grouping suggests a simple explanation of the “photon localization” recently observed by Casati et al. [1].

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1. Introduction

One of the most interesting discoveries to arise from an investigation of the quantum dynamics of a classically chaotic system is the localization of the quasienergy states of the periodically kicked rotor [2, 3]. This discovery immediately raised the question of whether such localization is a generic property of “Quantum Chaos”. Of particular interest is the relation between the classical diffusion rate and the quantum localization length in other non-integrable systems and whether this relation is universal. The present work is a numerical investigation of another canonical problem, the microwave excitation of hydrogen atoms. In particular, we investigate the nearest neighbour level spacing distribution and its relation to localization of the quasienergy states.

The generic behaviour of non-integrable, time-dependent systems is diffusive growth of the classical energy. When the Hamiltonian $H$ is composed of an integrable term $H_0$ and a term $H^p(t)$ that is periodic in time, the generic diffusive behaviour means that

$$\langle \Delta n^2 \rangle_{el} \sim D(n) t$$  (1)

where $n$ is the action of $H_0$ and $D$ is the diffusion coefficient. Depending upon the detailed relation between $H_0$ and the perturbation, the diffusion may occur globally or be bounded in the action-angle phase space.

When the Hamiltonian has period $T$, the quantum evolution is governed by the single period propagator $U(T)$ via

$$|\psi((N + 1)T)\rangle = U(T) |\psi(NT)\rangle$$  (2)

where $|\psi(0)\rangle$ is the initial state. $U(T)$ has quasienergy eigenstates $|\alpha\rangle$, and quasienergies [4] $\alpha$ which solve the eigenvalue problem

$$U(T) |\alpha\rangle = e^{i\alpha} |\alpha\rangle$$  (3)

where the quasienergies $\alpha$ are real since $U$ is unitary.

For Hamiltonians $H = H_0 + H^p(t)$ the long term behaviour of the energy will depend upon the form of the quasienergy states. If a quasienergy state extends over the entire action basis, then the action and the energy will grow with time. If, however, the quasienergy states have only a finite extent, then there will be a suppression of the diffusion.
This latter situation prevails in the best studied example known as the kicked rotor or Chirikov standard map [2, 3, 5–7]. In this model the classical energy follows the characteristic diffusive growth. However, the quasienergy states $\psi(n)$ are localized (except for non-generic resonance conditions) being of the form

$$
\psi_{n_0}(n) \sim e^{-\frac{|n-n_0|}{l(n_0)}}
$$

where the localization length $l(n_0)$ scales as $D(n_0)$. This means that an action state has significant overlap with only a finite number of quasienergy states. Thus, a rotor initially in a pure action state exhibits diffusion only for a short period of time. Diffusion stops when the quasienergy states have spread the occupation probability over their finite extent from which time on the energy displays complicated but bounded oscillations [2, 7].

Fishman et al. [7] have succeeded in mapping the rotor onto a one dimensional tight binding model with a hopping potential. They argue that the localization is due to the effective site energies of this model being random enough to cause Anderson localization [8]. It is an important open question whether localization and therefore suppression of diffusion is a universal feature of all quantum analogues of classically chaotic time periodic systems and whether Anderson localization is the only mechanism which accounts for the limitation of diffusion.

In order to discuss localization, it is useful to have an objective measure of the extent of quasienergy states. This is provided by the width function [9]

$$
W(n) = \exp\left(-\sum_s |\langle s|n\rangle|^2 \ln |\langle s|n\rangle|^2\right)
$$

which approaches the question from the point of view of the extension of action states in the quasienergy basis.

A signature of the chaotic nature of the classical dynamics may also be found in the energy or quasienergy spectrum. In any system the average density of states and its global behaviour depend upon the detailed dynamics, but the fluctuations about the average have a universal feature which reflects the classical dynamics. This feature is the clustering of levels of regular systems and the level repulsion of chaotic ones. The presence of level repulsion can be made manifest through various statistical measures [10] on the spectra obtained from laboratory or numerical experiments. Semiclassical treatments [11, 12] have succeeded in deriving some of these results for the energy spectra of conservative systems.

The statistic we will explore is the nearest neighbour level spacing (NNS). To define it we start with the spectral staircase

$$
N(E) = \sum_s \Theta(E - \alpha_s)
$$

and the spectral density

$$
d(E) = \frac{\partial N(E)}{\partial E} = \sum_s \delta(E - \alpha_s).
$$

The average staircase $\langle N(E) \rangle$ is the best smooth fit to $N(E)$ and the average density is

$$
\langle d(E) \rangle = \frac{\partial \langle N(E) \rangle}{\partial E}.
$$

The NNS is the probability distribution $P(s) ds$ of the normalized spacings $s_j$ between adjacent quasienergies where $s_j$ is defined as [13]

$$
s_j = \frac{\alpha_{j+1} - \alpha_j}{\sqrt{\langle \left(\alpha_{j+1} - \alpha_j\right)^2 \rangle}}.
$$

Our motivation for using the NNS is that when it is calculated for classically chaotic conservative systems the characteristic level repulsion shows up as $P(s) \rightarrow 0$ as $s \rightarrow 0$. This repulsion naturally depends upon there being significant interaction between all the eigenstates, a situation that may not hold for quasienergy states due to the localization phenomenon. Indeed, Feingold et al. [14] have found that the NNS of all the kicked rotor's quasienergies follows a Poisson distribution, but that if the quasienergies are selected from families of states having equal mutual overlap then some repulsion is evident. For a similar model consisting of a finite number of angular momentum states, Izrailev [15] has also obtained level repulsion.

This completes our introduction to the main issue of localization and the statistical method we shall use to explore it. In the next section we turn to the details of the model to be examined as well as a summary of the classical and quantum dynamics. The third section concerns the statistical analysis. After a brief description of the method, we present the results. A concluding section reviews the main findings.

2. The perturbed hydrogen atom

The physical system to be examined consists of hydrogen atoms prepared in states of high principal quantum number and passed through a microwave cavity operated at a single frequency. The proportion of atoms that have ionized during their passage through
the cavity is measured, the main result [16–24] being 
a sharp onset of ionization when the field strength 
\( \varepsilon \) exceeds a threshold value \( \varepsilon_t \). For a particular choice 
of field frequency and some principal quantum 
numbers, additional structure in the form of sub-
threshold bumps in the ionization versus field 
strength curves has been observed [19, 21] and 
explained [24].

Those individual atoms whose orbital angular 
momentum is small interact most strongly with the 
external field. The elongated form of their orbits 
allows the free atom to be modelled by the one dimen-
sional Hamiltonian [25, 26]

\[
H_0 = \begin{cases} 
\frac{p^2}{2} - \frac{1}{z} & z > 0 \\
\infty & z \leq 0 
\end{cases}
\] (10)

where \( p \) and \( z \) are the respective momentum and posi-
tion of the atomic electron within the one dimensional 
model. Atomic units are used in (10) and throughout 
this paper. The perturbation effected by the external 
microwave field adds a term to (10) to give

\[
H(z, t) = H_0 + \varepsilon z \sin(\omega t)
\] (11)

where \( \varepsilon \) and \( \omega \) are the field’s strength and frequency 
respectively. The validity of (11) as an approxima-
tion to the experimental system is discussed in [26, 27].
The model Hamiltonian (11) allows both bounded 
and unbounded (continuum) motion. In most prob-
lems of interest, the atom is initially prepared in a 
specific bound state, and the microwave field induces 
transitions which may finally lead to ionization. For 
weak fields near the ionization threshold, the rate of 
ionization is determined by the rate of upward transi-
tion in the bound space. Classically, upward transition 
leading to ionization is only possible when global 
chaos sets in, at which point the transfer of energy is 
diffusive. Quantum mechanically, the nature of the 
extcitation process is determined by the nature of the 
Floquet states (see below). To compare the quantum 
and classical excitation processes, it is sufficient to 
study the bound motion, and neglect the effect of the 
continuum on the bound space dynamics. This 
tuitive reasoning is supported by detailed calculations 
and considerations:

\[ i) \] In a previous publication [24] we have demo-
strated the high predictive power of bound space 
restricted models of the driven hydrogen atom. Experi-
mentally measured ionization thresholds e.g. can be 
reproduced without reference to the continuum.

\[ ii) \] We have shown [24] that only hydrogen 
bound states with principle quantum number \( n \) larger 
than \( n_w = \left( \frac{1}{3} \right)^{1/4} \) are coupled to the continuum 
whereas states with \( n < n_w \) are dynamically decoupled. 
For \( \omega n^4 > 1 \) classical diffusion and chaos in the bound 
space starts at [27] \( n = \left( \frac{1}{50 \pi^{1/3} n_w} \right)^{1/9} \) and since 
\( n < n_w \), quasienergy states whose support is mainly 
in the range of unperturbed states \( n < n_w \) (which 
later on will turn out to be the most interesting range) 
can safely be described without coupling to the con-
tinuum.

\[ iii) \] Quasienergy states that overlap mainly with 
unperturbed states whose principle quantum number 
is larger than \( n_w \) will be affected by the continuum. 
Using a Sturm-basis approach which takes the cou-
pling to and within the continuum into account, we 
have shown [24, 28] that the boundspace projected 
 quasienergy wavefunctions differ only in their tails 
from the boundspace wavefunctions calculated in the 
model without the continuum.

\[ iv) \] Finally, a bound space restricted model is 
interesting in its own right for comparison with the 
calculations of other authors [1, 27, 29] and as the 
rigorous mathematical analogue of classical diffusion 
models in action-angle space [30].

The bound space Hamiltonian may be written in 
action-angle variables \( n, \theta \) as

\[
H = -\frac{1}{2n^2} + \varepsilon z(n, \theta) \sin(\omega t)
\] (12)

\[
z(n, \theta) = \frac{3}{2} n^2 \left( 1 - \frac{4}{3} \sum_{k=1}^{\infty} x_k \cos(k\theta) \right)
\]

where \( x_k = \zeta_k(k)/k \).

The classical motion is then described by means of 
the Poincaré map

\[
n((N+1)T) = M^{\omega}(n(NT), \theta(NT)) \]
\[
\theta((N+1)T) = M^{\omega}(n(NT), \theta(NT))
\] (13)

obtained by integrating the equations of motion 
through one period \( T = \frac{2\pi}{\omega} \) of the perturbation. The 
Poincaré map derived (numerically) from (12) shows 
that the phase space of the bounded motion is divided 
into three regions [31]. When \( n \) is small, free atomic 
motion dominates and \( n \) is approximately a constant 
of the motion. When \( n \) is very large and \( \theta \) is not 
too close to either 0 or \( 2\pi \), \( z(n, \theta) \) is approximately 
constant on the time scale of the applied field. How-
ever, when \( \theta \) approaches 0 or \( 2\pi \) the effect of the 
atomic binding field becomes significant and \( z \) is no
The width function plotted in Fig. 1a shows a sharp transition between type I and III states characteristic of $\omega n^2 < 1$. There is a second regime $\omega n^2 > 1$, in which the transition is much more gentle (Fig. 1b). $\omega n^2 < 1$ means that there are no primary resonances between atomic states around $n$. In this regime the classical diffusion coefficient grows exponentially [31] with $-\frac{1}{\omega n}$ and so the width grows rapidly. In particular, it is meaningless to speak of a localization length scaling with the diffusion coefficient as is found for the rotor. When $\omega n^2 > 1$ Casati et al. [27] estimate the localization length as

$$l \approx D_{cl}(n) \approx \frac{2 \varepsilon^2 n^3}{\omega^{7/3}} = \frac{2 \varepsilon^2 n^3}{\omega^{7/3} n_0},$$

(14)

where $n_0$ is the initial state, $\omega_0 = \omega n_0^2$ and $\varepsilon_0 = \varepsilon n_0^3$. A consequence of this is that the width grows as $n^3$. The details of the analyses appropriate for the two regimes differ, as will be shown in the next section.

3. Numerical results

The numerical procedure used to obtain the quasienergy states and quasienergies was diagonalization of the one period propagator. The propagator itself was computed by dividing the period into time steps during which the field was assumed to have the constant value that it actually has only in the middle of the step. In all calculations reported here we used 80 time steps per period which was sufficient for convergence without being unduly expensive computationally. We also chose the first 180 atomic states as the basis in order to minimize finite size effects on the region of greatest interest, that of intermediate $n$.

The three regions revealed by the width function correspond to different types of classical behaviour. This can be made manifest by calculating a separate quasienergy NNS for each group of eigenstates. In order to do this, it is necessary to identify to which group a particular quasienergy state belongs. In some instances, an even more precise partition is useful, and to this end we define the maximum overlap.

The atomic state $|n\rangle$ is the eigenstate of (10) with eigenvalue $-\frac{1}{2n^2}$. The quasienergy state $|z\rangle$ is said to have maximum overlap with $|n(z)\rangle$ if

$$|\langle n(z) | z \rangle| = \max_n |\langle m | z \rangle|.$$

(15)

We can order the quasienergy states according to their maximum overlap and describe a state $|z\rangle$ as low lying (high lying) if $n(z)$ is small (large). Similarly,
we can sort the quasienergy states according to their maximum overlap with the $|z\rangle$ states. Thus, $|z(\alpha)\rangle$ denotes the $|z\rangle$ state having maximal overlap with $|z\rangle$. As we shall see, these orderings are also useful within regions.

We now present a detailed discussion of the NNS of the three regions of the width function, considering first the $\omega n^2 < 1$ regime.

In region I the quasienergy eigenfunctions $|\alpha\rangle$ correspond quite closely to unperturbed states $|n\rangle$. This means that the time dependent part of the Hamiltonian (11) does not affect these states and that they will evolve in time as if they were eigenfunctions of (10). We conclude that for those states $|\alpha\rangle$ such that $W(n(\alpha))=1$

$$\alpha \approx \frac{1}{2n(\alpha)^2} \frac{2\pi}{\omega} \mod 2\pi. \quad (16)$$

For later calculational convenience we shall express all quasienergies as fractions of $2\pi$ and this is henceforth assumed in (16) as well.

Figure 2 is a plot of $\alpha$ versus $\frac{1}{2n(\alpha)^2} \frac{1}{\omega} \mod 1$ for various values of $\omega$. The frequencies are chosen such that for any pair of frequencies $\omega, \omega'$, $\frac{1}{2n(\alpha)^2} \frac{1}{\omega} \mod 1 = \frac{1}{2n(\alpha)^2} \frac{1}{\omega'} \mod 1$. The plot shows negligible scatter about the diagonal which supports the contention that $n$ is an approximate constant of the motion. The modulus operation in (16) has the result of “randomizing” the sequence of quasienergies and distributing them evenly in the interval $[0, 1]$. The number of states in region I for a given frequency is small. In Fig. 3a we show the NNS of region I averaged over the frequencies listed in Fig. 2. This is allowed due to the pseudorandom nature of the quasienergies in region I. The resulting histogram follows very closely a Poisson distribution and in particular does not exhibit level repulsion.

Despite the fast rise in the width function, it is possible to separate type II states from type III in the $\omega n^2 < 1$ regime. We calculated the maximum overlap of the quasienergy states against both the atomic states and the $z$ states. Those whose overlap was less than a certain threshold value $g$ were deemed to be transitional, while the remainder were considered to be either of atomic or $z$ type. The spectral staircases and level densities were then calculated for the transition and $z$ type groups.

In the $\omega n^2 < 1$ regime the quasienergies of transition states are quite sensitive to $\varepsilon$ (Fig. 4) over a range of field strength that leaves the width function unchanged. The abundance of avoided crossings ensures that the spectra for two values of $\varepsilon$ differing by a small amount will be essentially independent. This independence allows us to improve the statistics by averaging over several level density histograms.

Figure 3c and d shows the level density in the transition region for two different values of $g$. The distribution is evidently intermediate between the Poisson and Wigner types in form. Further, as $g$ decreases and only states having a small overlap with either $z$ or atomic states are allowed, the magnitude of $P(0)$ decreases. Thus, the chaotic nature of this region of the classical phase space is reflected in correlations of the quasienergy spectrum and the correlations are stronger the more stringently we define transition states.

Figure 3b shows a level density histogram of states whose overlap with $z$ type states is greater than 0.7. The plot follows roughly the Poisson distribution. The plot suffers from a lack of statistics, which can only be improved at the price of decreasing the overlap threshold. We have done this and there is no essential difference in the quality of the plot. Note that the addition of a few correlated states will not affect the random nature of the nearest neighbour level density so long as the number of such additional states is small.

The Poisson nature of the plot is due to the same strong mixing mechanism as in region I. To see this, we calculate the quasienergies of region III to first order in perturbation theory. As the quasienergy states are asymptotically $|z\rangle$ states, we use the $|z\rangle$
Fig. 3. a NNS of region I, $\omega n_1^2 < 1$, averaged over the frequencies of Fig. 2, $\varepsilon \approx 3.74 \times 10^{-6}$. b NNS of region III, $\omega n_1^2 < 1$ regime, $\varepsilon \approx 3.79 \times 10^{-6}$, $\omega \approx 1.51 \times 10^{-6}$. Included are quasistability states $|z\rangle$ for which $|\langle z|z\rangle| > 0.7$. c Averaged NNS of region II, $\omega n_1^2 < 1$ regime, $\omega \approx 1.51 \times 10^{-6}$. Included are states for which both $|\langle z|z\rangle|$ and $|\langle z|n\rangle|$ are less than $\varepsilon = 0.1$ and the average is over 17 values of $\varepsilon$ lying between $3.713 \times 10^{-8}$ and $3.857 \times 10^{-8}$, d as e but for $\varepsilon = 0.7$. The dotted line is $e^{-\varepsilon^2}$ and the solid line is $e^{-\varepsilon^4}$.

Fig. 4. Dependence of the quasistabilities on the field strength $\varepsilon$ in the $\omega n_1^2 < 1$ regime, $\omega \approx 1.5985 \times 10^{-6}$. Level spectra for two values $\varepsilon_1$ and $\varepsilon_2$ of the field are considered statistically independent if multiple avoided crossings occur in the region $\varepsilon_1 < \varepsilon < \varepsilon_2$.

states as a zeroth order approximation. Working in the interaction representation, a straightforward calculation yields the first order quasistabilities as

$$ z = z_0(\alpha) T $$

where $|\alpha\rangle$ has maximum overlap with $|z(\alpha)\rangle$ and $z_0(\alpha) = \langle z(\alpha)|H_0|z(\alpha)\rangle$. Numerical evaluation of $z_0(\alpha)$ yields

$$ z_0(\alpha) \approx \frac{0.556}{(\varepsilon(\alpha) + \frac{1}{2})^2}, \quad \varepsilon \in N $$

so that

$$ z \approx \frac{0.556}{\omega (\varepsilon(\alpha) + \frac{1}{2})^2} \mod 1. \quad (19) $$

The eigenstate $|z(\alpha)\rangle$ has its maximum value $\langle n|z(\alpha)\rangle$ at roughly $n \approx z(\alpha)$. Substituting in (19) we arrive at our final estimate

$$ z \approx \frac{1}{2 \omega n(\alpha)^2} \mod 1. \quad (20) $$

For $\omega n_1^2 < 1$ and $n$ in the vicinity of $n_1$,

$$ \frac{1}{2 \omega n^2} \approx \frac{1}{2 \omega (n+1)^2} \approx \frac{1}{\omega n^2} > 1. \quad (21) $$

The strong mixing induced by the modula operation of (20) produces pseudorandomness as for the low lying states, and the NNS has a Poisson form.

We turn now to the $\omega n^2 > 1$ regime. The description of region I is identical to that of the $\omega n^1 < 1$ regime and there is nothing more to be learned.

Our description of region III may, in contrast, be considerably improved as we can employ the ordering of the quasistabilities implied by (20) and (21) to advantage. For the ranges of field strength $\varepsilon$ and driving frequency $\omega$ considered (dictated mostly by computational constraints), when $\omega n^2 > 1$, the integer part of

$$ \frac{1}{2 \omega n^2} $$

is small ($\leq 6$) for $n$ above the transition region.

If the perturbation theory is adequate, then the quasistabilities can be grouped according to the integer part of their premodulo values $E$. Within each group,
\[ \langle N(E) \rangle \text{ should go as } 1/\sqrt{E}. \text{ We call these groupings photons [1] and } \]

\[ p = 1 + \left\{ \frac{1}{2\omega n^2} \right\} \]  \hspace{1cm} (22)

the photon number, since an electron whose principal quantum number lies in the \( p \)\textsuperscript{th} grouping must absorb \( p \) photons in order to reach the continuum. In (22) \{ \} denotes the integer part.

We use expression (20) (without the modulus of course) to assign each high lying state to its photon and then calculate the spectral staircase, density and level spacing for each photon. The errors introduced in (20) are largely overcome by this procedure since only states near the borders of the photons might be incorrectly assigned. Within the photons the quasienergies “order themselves” and the constants are taken care of by the best fit procedure.

States whose premodulio quasienergies lie in the interval \([m, m+1]\) \( m=0,1 \) are fit to \( \langle N(E) \rangle = a/\sqrt{E} + m + b \). Figure 5 shows the spectral staircase and level spacing for these first two photons. The level spacing of the first photon is essentially a \( \delta \) function and there is some broadening and more noise in the second. The deviation is attributed to the greater effect of the free dynamics on the photons containing relatively lower lying states. Additionally, the photons are necessarily smaller so that the relative error introduced by misassigning states is greater. We
conclude that the quasienergy states of region III are strongly localized (in the z basis) but as a result of an (approximate) constant of the motion and not through the Anderson effect associated with random quasienergies.

It should be understood that the absence of photons in the $\omega n^2 < 1$ regime is due purely to limits on what was feasible computationally. Had it been possible to include very high lying states the photons would have appeared. Moreover, the good results obtained here confirm that it is indeed strong mixing of well determined eigenvalues that produces the pseudorandom distribution in the $\omega n^2 < 1$ regime.

When $\omega n^2 > 1$ the transition region is defined as those quasienergy states for which $1 < W(n(\alpha)) < n$. Figure 6 shows that $\langle N(E) \rangle$ is linear in $E$ and Fig. 7 shows that there is some level repulsion with $P(s=0)$ small. Although the histogram in Fig. 7 is not inconsistent with $P(s=0) = 0$, a finite value of $P(0)$ would not be unexpected since not all the quasienergy states in the transition region have the same localization length. There will be some states in the transition region that do not interact strongly and their quasienergies will be uncorrelated.

Assuming that the transitional states are exponentially localized (4) with localization length $l$ given by (14), the width function in the transitional region should grow as $n^2$. The straight line in Fig. 1b has a slope of 3, while a best fit to region II has a slope of 2.9. A different check on (14) was less conclusive. Keeping $\alpha_0$ and $\omega_0$ fixed and such that $n_0 = 42$ implies $\varepsilon \approx 5.65 \times 10^{-8}$ and $\omega \approx 6.08 \times 10^{-2}$, we varied $n_0$ between 28 and 49. We found that the exponent of $n_0$ lay between $-1.2$ and $-3.1$ as compared to the value $-1$ predicted by (14). We feel that the results of this particular check are hardly conclusive, but they certainly do not conflict with the estimate. More definite statements could be made from the higher values of $n_0$ which were beyond our computational reach.

An examination of the one cycle propagator of the $\omega n^2 > 1$ region provides an interesting perspective on the photon localization [1]. Figure 8a and c show that in the transition region ($25 \leq n \leq 60$) the magnitude of the diagonal and nearest neighbour terms of the propagator are comparable. In such a situation we might expect strong diffusion of the energy. However, as is known from the kicked rotor, if the elements of the propagator have pseudorandom phases localization can occur. In the kicked rotor, for sufficiently large $n$, pseudorandomness occurs on the scale of single unperturbed angular momentum states and so localization occurs on this same scale. Figure 8d shows the phases of the diagonal elements of the single period propagator plotted against $\frac{1}{2 \omega n^2}$. In the figure the phases corresponding to states within a single photon are connected by straight lines. Phases within a photon are correlated, but phases of states differing by more than a photon are uncorrelated, i.e. if locally a photon contains $L$ states, then locally the sequence of every $L$th phase is pseudorandom. Therefore, in contrast to the kicked rotor, we expect Anderson localization to occur on the scale of photons. The conclusion that the quasienergy states are localized in the photon number, $\psi \sim e^{-p}$, yields

$$p \sim \left( \frac{1}{n_0^2} - \frac{1}{n^2} \right) \approx \frac{2(n-n_0)}{n_0^2}$$

(23)
4. Conclusion

We have presented numerical evidence that the quasienergy states of the microwave perturbed hydrogen atom are of three types. The first type of state is one that is approximately an eigenstate of the unperturbed system. Such states are the quantum analogues of a region of regular motion, and their quasienergies are therefore predictable. However, because of the mixing induced by the purely numerical effect of the modulus operation, these quasienergies exhibit some of the features of a truly random sequence.

The second type of state is asymptotically an eigenstate of the dipole operator $\mathbf{\hat{z}}$. Here too the quasienergies are predictable, and depending on the details of the parameters, are either mixed or not by the modulus operation. In particular, when $\omega n_2^2 > 1$ the quasienergies show little fluctuation about their average value. Again, this is the expected behaviour for a region whose classical analogue displays approximately regular behaviour.

The third type of state is transitional between the first two types and is equally affected by both the free motion and the perturbation. These states, taken together, display level repulsion between their quasienergies. This differs somewhat from the kicked rotor where the states are so strongly localized that only subsets of the quasienergy states have a strong enough mutual interaction to induce level repulsion. We have found some indication that for the case of $\omega n_2^2 > 1$ the localization is due to a generalization of Anderson localization wherein the “site energies” are correlated over short distances corresponding to photons, but the photons themselves are not correlated. This supports the recent results of Casati et al. [1].

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References