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ABSTRACT

In the pseudo potential approximation the type of dynamics of two charged particles confined in a Paul trap depends on the ratio \( \lambda = \omega_z/\omega_r \) of the frequencies of the axial and the radial pseudo oscillators, respectively. For \( \lambda = 1/2, 1, 2 \) the classical dynamics is integrable. While there is no general proof, the classical dynamics appears to be nonintegrable for all \( \lambda \neq 1/2, 1, 2 \). For \( \lambda = \sqrt{2} \), e.g., we find a mixed phase space with interspersed regular and chaotic regions. Quantum mechanically, for \( \lambda = \sqrt{2} \) we find level repulsion whereas level clustering is observed for the integrable case \( \lambda = 2 \).

1. INTRODUCTION

In conjunction with modern laser cooling schemes\(^1\) the Paul trap\(^2-5\) has developed into one of the most useful and versatile tools in ion spectroscopy. In fact, its range of applicability is even wider: The Paul trap is proposed as the core of future frequency standards\(^6,7\) and it was used to test Weinberg's nonlinear generalization of quantum mechanics\(^8\). Aside from its practical value as a tool in ion spectroscopy, technology and fundamental physics the Paul trap has proved to be an interesting system in itself. Early experiments with charged aluminum particles\(^9\) demonstrated that particles confined in a Paul-trap can exhibit complicated "cloud like" behavior, but also regular "crystalline" behavior. It can be shown that within the framework of classical mechanics the clouds correspond to chaotic motion whereas the crystals correspond to integrable motion\(^10\). Thus, the
Paul trap is an excellent device for studying the classical nonlinear dynamics of few body systems.

By perfecting the technique of laser cooling it was shown recently that crystals and clouds can be observed on the atomic level.\textsuperscript{10–14} These experiments are more than just a repetition of the early experiments by Wuerker Shelton and Langmuir on the microscopic level. The demonstration of clouds and crystals on the atomic level is significant. It makes possible the study of quantum effects in a classically nonintegrable system. Indeed, quantum states of a single trapped particle have already been observed.\textsuperscript{15} This offers the opportunity to test existing quantum theories on single trapped ions.\textsuperscript{16–18} It may even be possible to extend these experiments to the many particle case in order to test quantum theories of more than one particle\textsuperscript{19} and to investigate the quantum manifestations of classical chaos in a Paul trap.

In this paper we study the quantum mechanics of two charged particles in the Paul trap. In order to facilitate the calculations we use the pseudo potential approximation. The relative motion of the two particles is formally equivalent to the quantized motion of a single particle in a two-dimensional potential and reminds of similar problems, such as, for instance, quantum billiards.\textsuperscript{20} Thus, the quantum manifestation of chaos in the Paul trap is expected to be rather similar to the quantum manifestation of chaos in billiards. In the case of billiards level repulsion and a Wignerian nearest neighbor level statistics was observed in the chaotic regime\textsuperscript{20} whereas level clustering and Poissonian statistics is expected for classically integrable quantum systems.\textsuperscript{21} Indeed, we will present evidence for level repulsion if the Paul trap is operated in a chaotic regime and level clustering if the two-particle motion is integrable.

The paper is organized in the following way: In order to introduce the subject, we will present some background material on the Paul trap in section 2. In section 3 we will motivate the use of the pseudo potential as an approximation to the one-cycle propagator of the full time dependent problem. With the help of an impulsively driven Paul trap we will demonstrate analytically that, at least in the single particle case, the pseudo potential is indeed an excellent approximation to the one-cycle propagator in the limit of weak trap fields. In section 4 we will present the results of numerical calculations of energy levels and their statistical characterization in an integrable ($\lambda = 2$) and in a nonintegrable case ($\lambda = \sqrt{2}$). In section 5 we will
discuss our results and indicate possible research directions. Also, we will discuss whether actual experiments can be performed to verify the quantum manifestations of classical chaos in a Paul trap. Section 6 will summarize our results and conclude the paper.

2. THE PAUL TRAP

In order to introduce the notation and to present some background material we will now briefly review the classical equations which govern the motion of charged particles in a Paul trap. The lowest order purely electric multipole field which allows for a stable confinement of charged particles is a time dependent quadrupole field which can be written as\(^{2-5}\)

$$\Phi(x, y, z) = \frac{U_0 + V_0\cos(\Omega t)}{r_0^2 + 2z_0^2} (x^2 + y^2 - 2z^2) . \tag{2.1}$$

The frequency of the field is denoted by \(\Omega\) and \(x\), \(y\) and \(z\) are the spatial coordinates. The classic Paul trap realizes this field with a set of three electrodes in the shape of axi-symmetric rotation hyperboles which follow the equipotential surfaces of (2.1). One of the electrodes, called the ring-electrode, is a single sheeted rotation hyperbola. Its distance from the axis of the trap is \(r_0\). The other two electrodes form the "end caps" of the trap. The end caps are electrically connected. Their distance of closest approach is \(2z_0\).

Stable confinement of charged particles in the potential (2.1) can be achieved for a range of dc and ac voltages applied between ring electrode and end caps. The corresponding voltages are denoted by \(U_0\) and \(V_0\), respectively. Because of Earnshaw's theorem, \(V_0 \neq 0\) is necessary for stable confinement. This condition, however, is not sufficient as stable confinement exists only for specific regions in \(U_0 - V_0\) space. These regions are called "stability regions". The Newtonian equations of motion of a single charged particle with mass \(m\) and charge \(e\) in the potential (2.1) is given by

$$m \frac{d^2}{dt^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + e \mathbf{\nabla} \Phi = 0 . \tag{2.2}$$

It is customary at this point to introduce the dimensionless control
parameters

\[ a = \frac{8eU_0}{m\Omega^2(r_0^2 + 2z_0^2)}; \quad q = \frac{4eV_0}{m\Omega^2(r_0^2 + 2z_0^2)} \]  \tag{2.3}

and the dimensionless time \( t' = \Omega t / 2 \) to turn the equations of motion into a set of three standard decoupled Mathieu equations\(^{22}\)

\[ \frac{d^2}{dt'^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \left[ a + 2q \cos(2t') \right] \begin{pmatrix} x \\ y \\ -2z \end{pmatrix} = 0 \]  \tag{2.4}

Equations of this type were first studied by E. Mathieu\(^{23}\) in the last century in connection with a vibrating membrane with an elliptic boundary.\(^{24}\) The important feature about (2.4) is that the differential equations are linear. This prohibits the occurrence of chaos in a single particle Paul trap which is of an exact quadrupole type. Higher order traps were recently considered.\(^{25}\) The single particle equations of motion for these traps correspond to driven nonlinear parametric oscillators which can exhibit chaos already on the single particle level.\(^{26}\) Chaos in the quadrupole trap is possible only if at least a second particle is loaded into the trap. In this case crystallization and chaos have been observed.\(^{10-14}\)

The method of averaging, originally due to Kapitza\(^{27,28}\) can be used to demonstrate stable confinement in the radio frequency Paul trap. According to (2.4) the force \( \vec{F} \) acting on the charged particle is the sum of a constant force, \( \vec{F}_c \), and a rapidly oscillating force, \( \vec{F}_{osc} \), with \( \vec{F} = \vec{F}_c + \vec{F}_{osc} \). The force \( \vec{F}_{osc} \) is responsible for the "micro motion" of trapped particles. It also produces a secular effect on the particles, a mean force which can be derived from the effective potential \( U_{eff}(x, y, z) \). This potential is given by

\[ U_{eff}(x, y, z) = \frac{1}{8} < F_{osc}^2 > = \frac{q^2}{4} \left( x^2 + y^2 + 4z^2 \right) \]  \tag{2.5}

The angular brackets denote a time average. The potential (2.5) is called the pseudo oscillator potential. The total time averaged force acting on the charged particles is now given by

\[ \vec{F} = \vec{F}_c - \nabla U_{eff} = - \begin{pmatrix} \mu_x^2 x \\ \mu_y^2 y \\ \mu_z^2 z \end{pmatrix} \]  \tag{2.6}
with
\[
\mu_\rho = \sqrt{a + \frac{q^2}{2}}, \quad \mu_\Omega = \sqrt{2(q^2 - a)}.
\]  
(2.7)

In the original units the pseudo oscillator frequencies are \(\omega_\rho = \mu_\rho \Omega/2\) and \(\omega_\Omega = \mu_\Omega \Omega/2\). It has to be emphasized that the pseudo oscillator frequencies obtained by the method of averaging are only approximate values, since the method of averaging itself makes use of small quantity expansions.\(^{26}\) The exact values of the pseudo oscillator frequencies can be computed using Floquet theory.\(^{22}\) An example of an explicit analytical calculation of the pseudo oscillator frequencies will be presented in section 3 for the special case of a square pulse drive.

As was mentioned above, chaotic motion in the Paul trap is possible only if more than one charged particles are simultaneously present in the trap. In the pseudo oscillator approximation the equilibrium separation of the two particles is given by\(^{10}\)
\[
d_0 = \left[\frac{2e^2}{m \pi \epsilon_0 \Omega^2 (a + q^2/2)}\right]^{1/3}
\]  
(2.8)

if crystallization occurs in the \(x-y\) plane. Choosing \(d_0\) as the unit of length, the two-particle equation is given by
\[
\frac{d^2}{dt^2} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} + \begin{pmatrix} \mu_\rho^2 x_i \\ \mu_\rho^2 y_i \\ \mu_\rho^2 z_i \end{pmatrix} = \frac{1}{2} \mu_\rho^2 \sum_{i \neq j} \frac{\vec{r}_{ij} - \vec{r}_{ij}}{|\vec{r}_{ij} - \vec{r}_{ij}|^3}, \quad i = 1, 2
\]  
(2.9)

with \(\vec{r}_i = (x_i, y_i, z_i)\). The equations (2.9) decouple in relative and center of mass coordinates. Defining \(\vec{r} = \vec{r}_1 - \vec{r}_2\) and introducing cylinder coordinates \((r_x = \rho \cos \phi, \quad r_y = \rho \sin \phi, \quad r_z = \zeta)\) the equations of motion for \(\rho\) and \(\zeta\) in the pseudo potential approximation can be stated as\(^{10}\)
\[
\ddot{\rho} = \frac{\nu^2}{\rho^3} - \rho + \frac{\rho}{(\rho^2 + \zeta^2)^{3/2}}; \quad \ddot{\zeta} = -\lambda^2 \zeta + \frac{\zeta}{(\rho^2 + \zeta^2)^{3/2}}.
\]  
(2.10)

In equations (2.10) the differentiations are with respect to scaled time \(\tau = t' \mu_\rho\) and \(\nu\) is the scaled angular momentum given by \(\nu = \rho^2 \Phi / \mu_\rho\). Besides \(\nu\) the equations (2.10) contain a second control parameter \(\lambda\) which is defined as \(\lambda = \mu_\Omega / \mu_\rho = \omega_\Omega / \omega_\rho\).
Defining the momenta $p_\rho = \dot{\rho}$ and $p_\zeta = \dot{\zeta}$ the equations of motion (2.10) can be derived from the Hamiltonian

$$H = \frac{1}{2} p_\rho^2 + \frac{1}{2} p_\zeta^2 + V(\rho, \zeta)$$  \hspace{1cm} (2.11)

with

$$V(\rho, \zeta) = \frac{1}{2} \rho^2 + \frac{1}{2} \lambda^2 \zeta^2 + \frac{\nu^2}{2 \rho^2} + \frac{1}{(\rho^2 + \zeta^2)^{1/2}} \hspace{1cm} (2.12)$$

Repeating steps (2.1) - (2.12) on the quantum level it is easy to show that the quantized version of (2.11) is given by

$$\hat{H} = -\frac{\alpha^2}{2} \left\{ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \frac{1}{\rho^2} \frac{d^2}{d\Phi^2} + \frac{d^2}{d\zeta^2} \right\}$$

$$+ \frac{1}{2} \rho^2 + \frac{1}{2} \lambda^2 \zeta^2 + \frac{1}{(\rho^2 + \zeta^2)^{1/2}}$$  \hspace{1cm} (2.13)

where

$$\alpha = \frac{2\hbar}{m\omega_\rho d_0^2}$$  \hspace{1cm} (2.14)

plays the role of an effective dimensionless $\hbar$. The energy eigenfunctions are denoted by $\psi_n$ so that $\hat{H}\psi_n = E_n \psi_n$. Substituting, in the standard way, $\varphi = \sqrt{\rho}\psi$ and separating $\varphi = \tilde{\varphi}(\rho, \zeta) \exp(iM\phi)$, the eigenvalue equation for $\psi$ is transformed into an eigenvalue equation for $\tilde{\varphi}$ according to $\tilde{\hat{H}}\tilde{\varphi}_n = E_n \tilde{\varphi}_n$ with

$$\tilde{\hat{H}} = -\frac{\alpha^2}{2} \left\{ \frac{d^2}{d\rho^2} + \frac{d^2}{d\zeta^2} \right\} + V(\rho, \zeta)$$  \hspace{1cm} (2.15)

and $\nu^2 = \alpha^2[M^2 - 1/4]$.

This concludes our basic review of the Paul trap. In the next section we will present a brief discussion of the interpretation of the quantum Hamiltonian (2.13).

3. THE SQUARE PULSE DRIVEN PAUL TRAP

In this section we show that in the single particle case the time averaged classical (pseudo) Hamiltonian is closely related to the quantum mechanical quasi energy operator. In order to gain as much
insight as possible, we will, restricted to this section, briefly aban-
don the sinusoidal drive and consider the square pulse driven Paul trap.\(^{50}\) For the square pulse, and in the single particle case, all the
necessary calculations can be done analytically so that the connection
between exact classical, exact quantum mechanical and average clas-
sical methods can be explored. Moreover, we will restrict ourselves to
the one-dimensional case and set the trap parameter \(a\) equal to zero.

The square pulse driven Paul trap is obtained by replacing the
cos-drive term in (2.1) by \(\cos(x) \rightarrow \theta(x)\) with \(\theta(x + 2\pi) = \theta(x)\) and

\[
\theta(x) = \begin{cases} 
1, & \text{for } 0 \leq x < \frac{\pi}{2} \\
-1, & \text{for } \frac{\pi}{2} \leq x < \frac{3\pi}{2} \\
1, & \text{for } \frac{3\pi}{2} \leq x < 2\pi
\end{cases} \quad (3.1)
\]

Expanding \(\theta(x)\) in a Fourier series and using the method of averaging
we obtain the pseudo oscillator potential

\[
U_{\text{eff}}(x) = \frac{1}{\pi^2} m \Omega^2 q^2 x^2 \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^4} \quad (3.2)
\]

The sum in (3.2) is known analytically. It is a special case of the
\(\lambda\)-function which is related to Riemann's \(\zeta\)-function\(^{22}\). With\(^{22}\)

\[
\sum_{n=0}^{\infty} \frac{1}{(2n + 1)^4} = \lambda(4) = \frac{\pi^4}{96} \quad (3.3)
\]

we have

\[
U_{\text{eff}}(x) = \frac{\pi^2}{96} m \Omega^2 q^2 x^2 \quad (3.4)
\]

From this expression we obtain the pseudo oscillator frequency

\[
\omega^{(\text{sec})} = \frac{\pi}{4\sqrt{3}} \Omega q \quad (3.5)
\]

Since the method of averaging requires that the amplitude of the micro
motion is small compared with the amplitude of the secular motion,
we expect that the result (3.5) is correct to first order in \(q\).

Next, we solve the classical problem of a single particle in a
square-pulse driven Paul trap exactly. Due to the discrete nature
of the square pulse, the equations of motion can be integrated in the
form of a mapping. This mapping is linear in the positions and momenta and given by

\[
\begin{pmatrix} x' \\ p' \end{pmatrix} = M \begin{pmatrix} x \\ p \end{pmatrix}
\]

(3.6)

where

\[
M = \begin{pmatrix} \gamma c & \frac{1}{m \omega_0} (\gamma s + \sigma) \\ -m \omega_0 (\gamma s - \sigma) & \gamma c \end{pmatrix}
\]

(3.7)

The constants in (3.7) are \( \gamma = \cosh(\nu) \), \( \sigma = \sinh(\nu) \), \( c = \cos(\nu) \), \( s = \sin(\nu) \), \( \omega_0 = \Omega(q/2)^{1/2} \) and

\[
\nu = \pi \sqrt{q/2}
\]

(3.8)

Since the mapping (3.6) was derived from a Hamiltonian, its determinant is 1. The mapping corresponding to a pure harmonic oscillator with frequency \( \omega \) over the time interval \( T = 2\pi/\Omega \) is given by (3.6) with \( M \to N \) and

\[
N = \begin{pmatrix} \cos(\omega T) & \frac{1}{m \omega} \sin(\omega T) \\ -m \omega \sin(\omega T) & \cos(\omega T) \end{pmatrix}
\]

(3.9)

Evidently, (3.7) is of this form and we obtain the exact classical pseudo oscillator frequency from (3.7) via

\[
\omega^{(cl)} = \omega_0 \left[ \frac{\gamma s - \sigma}{\gamma s + \sigma} \right]^{1/2}
\]

(3.10)

Expanding the right hand side of (3.10) in powers of \( q \) yields

\[
\omega^{(cl)}(q) = \frac{\pi}{4\sqrt{3}} \Omega q + o(q^2)
\]

(3.11)

As expected, this result agrees with the result (3.5) to lowest order in \( q \).

The next step is a quantum solution for the square pulse driven Paul trap. Since the drive is time periodic, we construct the one-cycle propagator for the square pulse. Since in all subintervals of one cycle of the square pulse the Hamiltonian corresponds to a (possibly inverted) harmonic oscillator, the total one-cycle propagator is that of a harmonic oscillator as well. The oscillator frequency \( \omega^{(qm)} \) equals the classical frequency (3.10) and can be calculated from

\[
\omega^{(qm)}(q) = \omega_0 \frac{\sigma}{[\varphi^2(\nu) - 1] \sin^2(\nu/2) [1 - A^2(\nu)]^{1/2}}
\]

8
\[ A(\nu) = \frac{1}{\sinh(\nu)} \left\{ \frac{1}{2} \sin(\nu)[\vartheta^2(\nu) - 1] - \vartheta(\nu) \sinh(\nu) \right\} \]

\[ \vartheta(\nu) = \left[ \coth(\nu) + \cot \left( \frac{\nu}{2} \right) \right] \sinh(\nu) \quad (3.12) \]

with \( \nu \) defined in (3.8). Expansion of \( \omega^{(qm)} \) in powers of \( q \) yields

\[ \omega^{(qm)}(q) = \frac{\pi}{4\sqrt{3}} \Omega q + o(q^2) \quad (3.13) \]

which agrees to lowest order with both, \( \omega^{(av)} \) and \( \omega^{(cl)} \).

Thus we have established that in the single particle case and for a square pulse drive the quasi energy operator and the time averaged and the classical pseudo Hamiltonians agree to first order in \( q \).

Motivated by this result we are confident that the eigenstates of (2.11) (or (2.15), respectively) are good approximations to the quasi energy states of the one-cycle propagator of the two-particle cos-driven Paul trap.

We will now turn to an investigation of the spectrum of (2.15) and suggest that the Paul trap is an excellent tool for testing theories on the spectral statistics of classically chaotic systems.

4. QUANTUM CALCULATIONS

The Hamiltonian (2.15) is symmetric in \( \zeta \). Therefore, a \( \zeta \)-parity can be introduced. The \( \zeta \)-parity can also be used to ensure the correct exchange symmetry in the case of identical particles. Concentrating on \( M = 0 \) we calculated numerically the energy spectrum \( E_n \) of (2.15) for \( \lambda = 2, \lambda = \sqrt{2}, \alpha = 0.01 \) and positive \( \zeta \)-parity. For \( E_n \leq 1.8 \) the result is shown in Fig. 1 in the form of a stair case function. Also shown, as the smooth solid line, is the mean stair case function obtained by fitting a quadratic polynomial to the exact stair case. The stair cases for \( \lambda = 2 \) (Fig. 1a) and \( \lambda = \sqrt{2} \) (Fig. 1b) look very different. This is not surprising since \( \lambda = 2 \) corresponds to an integrable situation\(^{10}\) whereas for \( \lambda = \sqrt{2} \) the phase space shows regular and chaotic regions. The nearest neighbor level spacing statistics corresponding to Figs. 1a and 1b are shown in Figs. 2a and 2b, respectively. Clearly, Fig. 2a shows level clustering whereas Fig. 2b shows level repulsion. This is the expected behavior for regular and chaotic systems,
FIGURE 1. Stair case function of the energy levels $1.5 < E < 1.8$ of the Paul trap in pseudo potential approximation. (a) The integrable case $\lambda = 2$, (b) the case with a mixed classical phase space $\lambda = \sqrt{2}$. In both cases $\alpha = 0.01$.

FIGURE 2. Nearest neighbor spacing statistics for the two cases shown in Fig. 1, respectively.
respectively. The smooth curves in Fig. 2 show the functions corresponding to Poissonian and Wignerian level statistics, respectively. These lines are drawn merely to guide the eye. We do not imply that in the energy regime \( E \leq 1.8 \) the integrable case \( \lambda = 2 \) would follow exactly Poissonian statistics and the "chaotic" case \( \lambda = \sqrt{2} \) would exhibit Wignerian statistics. For \( \lambda = \sqrt{2} \) the reason is that the phase space contains large regular regions for energies \( E \leq 1.8 \) (compare Fig. 3 in ref. 10), and (approximate) Wignerian statistics is expected only if the size of the regular regions is negligibly small, or regular regions are completely absent (the hyperbolic case). In the regular case \( (\lambda = 2) \) we do not expect to see Poissonian statistics for \( E \leq 1.8 \) because of (approximate) accidental degeneracies which are present in this energy region. These degeneracies are clearly visible in the stair case shown in Fig. 1a. The degeneracies disappear at around \( E > 1.8 \) where the nonlinearities in the pseudo potential take over.

Clearly our calculations have to be improved in order to access the energy region \( E > 1.8 \). But even in the present stage of our calculations we can say that regular motion in the Paul trap is indicated by level clustering whereas the signature of chaotic motion in the Paul trap is level repulsion.

5. DISCUSSION

In the previous section evidence was presented for level clustering and level repulsion corresponding to regular and chaotic motion in the Paul trap, respectively. Two obvious questions will be discussed in this section: (i) how to improve the calculations and (ii) whether experiments can be designed to detect the signatures of regular and chaotic motion in the Paul trap.

(i) Currently our calculations are limited to an effective Planck's constant \( \alpha \geq 0.01 \) which results in about 80 levels below \( E = 1.8 \) for \( \lambda = 2 \) and in about 140 levels below \( E = 1.8 \) for \( \lambda = \sqrt{2} \). We plan to push our calculations to effective \( \hbar \) values \( \alpha \ll 0.01 \) which results in more levels below \( E = 1.8 \) and accompanying better statistics. Also, it would be interesting to study the transition between chaotic and regular motion in the vicinity of \( \lambda = 2 \) and calculate the "width" \( \Delta \lambda \) of the transition. Around \( \lambda = \sqrt{2} \) where chaos prevails, better statistics can be produced by sampling the system at several different \( \lambda \) values around \( \lambda = \sqrt{2} \). If the steps in \( \lambda \) are large enough
so that every quantum level below $E = 1.8$ encountered at least one (avoided) crossing as a function of $\lambda$, the samples can be treated as approximately independent and can be added up for better overall statistics.

(ii) In order to select and identify the quantum states of charged particles in a Paul trap it is probably advantageous to operate the trap in a “stiff” regime, i.e., the focussing forces of the trap should be as large as possible. This results in a spreading out of the energy levels since the quantum particles will be confined to a smaller volume. Unfortunately the Paul trap cannot be operated with arbitrarily high trapping potentials since the Paul trap turns unstable at the so-called Mathieu instability. But even for the most favorable choice of $a$ and $q$ values, the effective $\hbar$ for Mg ions, e.g., is so small that tens of thousands of states are expected in the energy region $E \leq 1.8$. The level spacings are correspondingly small and experiments with heavy ions, such as Mg ions, are therefore very hard to perform. Electrons, on the other hand, result in much larger $\alpha$ values. This is a welcome improvement over the Mg case as far as an experimentally resolvable level density is concerned. On the other hand, when working with electrons one loses the possibility of laser cooling to access the low temperatures needed to enter the quantum regime.

6. SUMMARY AND CONCLUSIONS

In this paper we demonstrated that the quantum signatures of order and chaos in the Paul trap are level clustering and level repulsion, respectively. Our calculations were performed in the pseudo potential approximation which leaves room for future improvements such as, e.g., including the time dependence of the trap potential. We presented arguments to the effect that the pseudo potential is a good approximation to the time independent quasi energy operator. Therefore, explicit inclusion of the time dependence is likely to result only in minor modifications of the classical pseudo potential. An experimental demonstration of level clustering and level repulsion, however, seems to be very hard. In order to access the chaotic regime, one has to go to moderately high excitation energies since otherwise the quantum particles would feel only the quadratic minimum of the pseudo potential which can only result in regular motion. On the other hand when working with Mg ions, even at moderately high energies one is
confronted with thousands of quantum states which will probably be very hard to separate experimentally. Electrons are more promising in this respect, but have the disadvantage of being inaccessible to laser cooling. Nevertheless, we hope that experiments will soon be able to access this promising quantum regime.

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