Stretched Helium: A Model for Quantum Chaos in Two-Electron Atoms

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Abstract

We show that the one-dimensional model of helium (stretched helium) is a chaotic scattering system. We provide evidence for the existence of a repetitive sequence of three dynamical regimes as a function of the excitation energy of the model atom. The three dynamical regimes are: Ericson, Wigner and Rydberg. They are characterized by $\Gamma \gg s$, (Ericson), $\Gamma \lesssim s$ (Wigner) and $\Gamma \ll s$ (Rydberg), respectively, where $\Gamma$ is the width of the autoionizing resonances and $s$ is their mean (local) spacing.

1. Introduction

The three-body system of celestial mechanics is undoubtedly one of the outstanding unsolved problems in theoretical physics\textsuperscript{1).} For now more than two hundred years it has plagued physicists and mathematicians alike. The importance of the problem can hardly be over stated. Our own solar system can be considered as an assembly of three-body systems, the most important such sub-system, to us anyway, being the Earth-Moon-Sun system. With Newton's work as a starting point, many astronomers and eminent mathematicians tried to solve the three-body problem analytically. Despite enormous progress in the course of the last two centuries no real breakthrough occurred. It was Poincaré at the end of the last century\textsuperscript{2) who discovered the reason for the insolubility of the three-body problem:
chaos. This fact was like a jolt in the physics community. It means that in general (apart from well-known special solutions like Lagrange’s equilateral triangle solution or Euler’s collinear solution) the three-body problem cannot be solved analytically\textsuperscript{3}.

It is interesting to point out that a conceptually similar development occurred in the theory of parallels in the mathematical discipline of plane geometry. After millenia of attempts to prove Euclid’s fifth axiom, the impossibility of such a proof was demonstrated by Lobachevsky, Bolyay and Gauss. In mathematics this seemingly “negative” result opened the door for fruitful research in non-Euclidean geometry. Therefore, we should not be discouraged by Poincaré’s result. It marks the beginning of a fascinating subject: the investigation of deterministic chaos in three-body systems.

Chaos is not confined to the motion of celestial bodies. Slowly at first, but like an explosion during the last twenty years chaos was identified in nearly every corner of science. Chaos is everywhere. Not even atomic physics escaped the grip of chaos\textsuperscript{4}. Traditionally thought to be “protected” from chaos by the mitigating effects of wave mechanics\textsuperscript{5–11}, it was only recently appreciated that multiply excited atoms behave like strongly correlated miniature solar systems which clearly show chaotic effects.

In this article the helium atom will serve as a paradigm of a chaotic three-body system in atomic physics. It will turn out that chaos theory – classical as well as quantum mechanical\textsuperscript{12} – is the natural tool for a thorough discussion of many-electron atoms in the semiclassical regime. Not only does this theory help in the characterization of spectra and wave functions, it also makes specific predictions about the existence of new dynamical regimes\textsuperscript{13}. Even thermodynamic concepts will find their way into the atomic theory of three-body systems as chaos theory predicts the existence of a “gas of resonances”\textsuperscript{13,14}.

In order to get acquainted with the dynamical systems view of atomic physics, it is not necessary to study the three-dimensional helium atom. It was shown that a one-dimensional version of the helium atom captures many essential features of the complete three-dimensional atom\textsuperscript{13,15}. Treating a one-dimensional model enables us to focus on the dynamics of the system without being hampered by complications arising from angular momentum and fine structure. Although, admittedly, much physics is lost in reducing the helium atom to one dimension it is surprising how much physics is in fact preserved. In
this respect the one-dimensional helium atom behaves like the one-dimensional model of hydrogen atoms in a strong microwave field\textsuperscript{16–18}. The one-dimensional hydrogen atom captures much of the physics of the three-dimensional experiments\textsuperscript{19,20}. Surprisingly, a comparison between hydrogen ionization experiments and the predictions of the one-dimensional model makes sense even on the quantitative level\textsuperscript{21}. In fact, it was shown recently that the one-dimensional model works in the case of photo ionization of helium in a strong laser field\textsuperscript{22}.

In this spirit the paper is organized in the following way. In section 2 we present the one-dimensional model of the helium atom, the “stretched helium” model. Recently much research work was focussed on the investigation of this model\textsuperscript{13,15,23–26}. Some aspects of the classical and quantum mechanics of stretched helium are reviewed briefly in section 3. An interesting view of stretched helium as a chaotic scattering system is presented in section 4. In section 5 we present evidence for the existence of three quantum regimes as a function of excitation energy. Section 6 concludes the paper.

2. The Model

One of the first experiments in atomic physics to show the importance of chaos on the atomic level were the microwave ionization experiments by Bayfield and Koch\textsuperscript{19}. Interpreting these experiments on the basis of perturbation theory turned out to be near impossible since typically multiphoton processes on the order of 100 have to be considered. On the other hand Leopold and Percival were the first to show that a classical Monte Carlo approach\textsuperscript{27} was successful in reproducing experimentally obtained ionization thresholds. Soon it was demonstrated\textsuperscript{28,29} that on the classical level chaos occurs in this system, and ionization thresholds were successfully interpreted as chaos thresholds\textsuperscript{16–18,20,21,28,30}. To our knowledge this was the first time when a dynamical systems point of view added to our understanding of a strictly quantum strictly atomic physics system. Moreover, it was demonstrated that in the microwave ionization system a full three-dimensional treatment is not necessary to obtain qualitative and even semiquantitative results. The one-dimensional “surface state electron” Hamiltonian\textsuperscript{16,17,29}

\[
h(x, p; Z) = \begin{cases} \frac{p^2}{2} - \frac{Z}{x} & \text{for } x > 0 \\ \infty & \text{for } x \leq 0 \end{cases} \quad (2.1)
\]
is sufficient to develop insight into the microwave ionization problem.

The simplest time independent atomic physics system capable of exhibiting chaos on the classical level is the helium atom. In analogy to the driven hydrogen atom briefly discussed above, it is enough to consider a one-dimensional version of the helium atom in order to introduce the dynamical systems point of view of the helium atom. A one-dimensional model of helium \((Z = 2)\) can be constructed by “glueing” two one-dimensional hydrogen atoms together, joining them at the origin which is occupied by the nucleus of the helium atom:

\[
H' = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{Z}{x_1} - \frac{Z}{x_2} + \frac{1}{x_1 + x_2} .
\] (2.2)

The situation is illustrated in Fig. 1. Electron 1 is confined to the left half space, electron 2 is confined to the right half space. We define \(x_1, x_2 > 0\) so that electron 1 is in fact located at \(-x_1\). This way the spatial dynamics of the two electrons takes place in the first quadrant of the \(x_1, x_2\) plane. Singlet and triplet states of helium correspond to symmetric and antisymmetric wavefunctions, respectively. We will work in \(Z\) scaled coordinates \(x \rightarrow x/Z\), \(p \rightarrow pZ\). This transformation yields:

\[
H = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{1}{x_1} - \frac{1}{x_2} + \frac{\epsilon}{x_1 + x_2}
\] (2.3)

with \(H = H'/Z^2\) and \(\epsilon = 1/Z\). For \(\epsilon \neq 1/2\) this Hamiltonian describes the isoelectronic sequence of two-electron atomic ions. In order to construct eigenstates of (2.3) it is useful to consider the eigenstates of the single particle Hamiltonian (2.1). It possesses a Rydberg series of bound states

\[
\varphi_n(x) = \langle x \mid n \rangle = n^{-3/2}(\frac{2x}{n}) L_{n-1}^{(1)}(\frac{2x}{n}) \exp(-x/n) ,
\] (2.4)

and a continuum which consists of the states

\[
\varphi_k(x) = \langle x \mid k \rangle = \frac{2kx}{\sqrt{k(1-e^{-2\pi/k})}} e^{-ikx} _1F_1(1 + \frac{i}{k}, 2; 2ikx) .
\] (2.5)

They can be used to construct the eigenstates of (2.3). Forming product states from (2.4) and (2.5), symmetric and antisymmetric bound-bound, bound-continuum and continuum-continuum states can be constructed which diagonalize (2.3) for \(\epsilon = 0\). Although the
Hamiltonian (2.3) looks rather innocuous, its dynamics is far from trivial. Many researchers produced evidence\textsuperscript{13,15,24–26} that on the classical level the Hamiltonian (2.3) is completely chaotic for $\epsilon \neq 0$. It appears to be integrable only for $\epsilon = 0$. In the integrable case the classical equations of motion for $x_1, p_1, x_2, p_2$ can be integrated with the help of a simple canonical transformation

$$\theta_i = 2\eta_i - \sin(2\eta_i) , \quad \frac{d\theta_i}{d\eta_i} = 4 \sin^2(\eta_i) ; \quad 0 \leq \eta < \pi$$

$$x_i = 2n_i^2 \sin^2(\eta_i) ; \quad p_i = \frac{1}{n_i} \cot(\eta_i) , \quad i = 1, 2 \quad (2.6)$$

Since the problem seems to be completely chaotic for $\epsilon \neq 0$ no such canonical transformation can exist in this case. We have to resort to numerical methods to integrate the equations of motion. Expressing the dynamical variables $x_1, p_1, x_2, p_2$ in terms of the action-angle variables $n_1, \theta_1, n_2, \theta_2$ defines the “bound space projected” approach\textsuperscript{31} to the one-dimensional helium problem. This is so because the action and angle variables $n$ and $\theta$ are only defined for bounded motion. The quantum analog of this classical projection method is to work in a Hilbert space which is spanned by products of the square normalizable states (2.4) only. Since this procedure defines an exact correspondence between the classical and the quantum models we will use this approach in the rest of the paper. It has to be emphasized that a basis constructed from the states (2.4) is not complete. The calculation of resonance widths, for instance, is beyond the scope of this approach. We found, however, that although the “bound space projected” basis (2.4) is incomplete, it yields very accurate results for the locations of auto-ionizing resonances as long as their widths are small. This condition is met for the energy regime investigated in section 5 where we consider the transition between the regular regime at low energies and the chaotic regime at moderately high excitation energies. In this regime the ratio of the decay widths and the separation of resonance energies is small\textsuperscript{13}. Thus, the bound space projected approach is justified. Moreover, the bound space projected calculations take only a fraction of the time required by calculations in a complete $L^2$ basis\textsuperscript{13}. Therefore, we felt that the bound space projected approach is adequate as a method for providing support for the ideas developed in this paper.
3. Brief review of stretched helium results

It is quite possible and we fully expect that the one-dimensional helium atom will acquire the same status as a paradigm in many electron atoms as was already attained by the one-dimensional hydrogen atom in microwave ionization\(^{18}\) or the kicked rotor\(^{5-9,11,32}\) in dynamical localization. Therefore, it is useful to explore the stretched helium model in every detail such that the results are handy for drawing qualitative analogies or pointing out discrepancies with the real three-dimensional helium atom. Although many results are known by now, much remains to be done. Especially as far as analytical proofs are concerned for properties of this system which by now can only be asserted on numerical grounds. One such property is the absence of any regular islands in the classical phase space. Although by no means established analytically, there is currently no evidence for the existence of any regular islands\(^{13,15,26}\). Therefore it is conjectured that the one-dimensional stretched helium atom is fully chaotic\(^{13,15,26}\). How dangerous such a claim may be was demonstrated recently by Dahlqvist and Russberg\(^{33}\) who were able to identify a small stable island for the \(x^2y^2\) problem previously conjectured to be fully chaotic by many authors. On the other hand the absence of stable islands was recently proved analytically for a system closely related to the present problem: the kicked one-dimensional hydrogen atom\(^{34,35}\). It is therefore not entirely out of the question to prove or disprove the chaoticity of stretched helium analytically.

Results on the periodic orbit structure of stretched helium seem to corroborate the chaoticity claim. All periodic orbits found so far are unstable and can be enumerated one-by-one by a binary code\(^{13,15,24-26}\). The remarkable feature of the periodic orbit structure is that no pruning is necessary. All imaginable finite binary sequences seem to correspond to a periodic orbit and vice versa. The one-to-one property of this mapping was proved to hold for all sequences up to length six\(^{26}\).

The periodic orbits were also used to perform a semiclassical quantization of stretched helium\(^{26}\) on the basis of Gutzwiller's formula\(^{12,36,37}\). Excellent results were reported using only a few periodic orbits and a special resummation technique\(^{26}\).

Excitation energies and widths were also calculated recently within the framework of a complex rotation \(L^2\) approach\(^{13}\). Results on real and imaginary parts of resonances
were obtained up to the tenth ionization threshold. It was established that the resonances corresponding to the eighth and higher thresholds move in the complex plane as a function of $\epsilon$ (see (2.3)) not unlike the particles in a highly correlated gas or liquid\textsuperscript{13).} We found indications that for increasing excitation energy the gas expands into the complex plane away from the real axis. Thus, for sufficiently high excitation energy a condition may be reached where the separation of the real parts of the resonance energies are comparable with their imaginary parts. This condition is expected to occur at energies around the 30'th ionization threshold and corresponds to a situation of overlapping resonances. In this case scattering cross sections are expected to exhibit wild oscillations which cannot be associated any more one-to-one with individual resonances. This condition was first encountered in nuclear physics in heavy ion collisions at relatively high excitation energies. The phenomenon is nowadays referred to as “Ericson fluctuations” since it was predicted to occur by T. Ericson in the sixties\textsuperscript{38,39).} Recently it was established that a system which exhibits chaotic scattering on the classical level is likely to show Ericson fluctuations in the semiclassical regime\textsuperscript{40–44).}

In the next section we will show that chaotic scattering indeed occurs in the stretched helium model. In section 5 we will establish the existence of a third dynamical regime between the regular and the Ericson regimes. It is characterized by relatively small resonance widths but strongly correlated resonance energies which leads to level repulsion.

4. Stretched Helium: A Chaotic Scattering System

In order to establish the connection between stretched helium and chaotic scattering, we first consider the potential energy in (2.3):

\[ V(x_1, x_2) = -\frac{1}{x_1} - \frac{1}{x_2} + \frac{\epsilon}{x_1 + x_2} \]  

(4.1)

Some equipotential lines for (4.1) are shown in Fig. 2 for $\epsilon = 1/2$. This potential reminds strongly of “elbow” scattering systems recently investigated analytically, numerically and experimentally\textsuperscript{45) in connection with chaotic scattering.

The potential shown in Fig. 2 clearly exhibits two asymptotic entry (exit) channels at $x_1 \to \infty$ and $x_2 \to \infty$, respectively. Scattering potentials of the type shown in Fig. 2
also occur in the theoretical description of chemical reactions. Since complicated dynamics and chaotic scattering was reported to occur in these systems\textsuperscript{46–48} it is natural to expect chaotic scattering to occur in the potential (4.1). In order to prove this point we launched 800 scattering trajectories at \( n_1 = 1, \quad \theta_1 = j \cdot (2\pi/801), \ j = 1, 2, \ldots, 800, \ x_2 = 10 \) and \( p_2 = -0.5 \). For all these trajectories we calculated the time \( T \) it takes to reach the "asymptotic" region which we defined to be \( x_1 \geq 10 \) or \( x_2 \geq 10 \). The result is shown in Fig. 3. The delay times \( T \) vary smoothly with the initial phase \( \theta_1 \) for \( 0 < \theta_1 < 4.5 \). At around \( \theta_1 \approx 4.55 \) we see a cusp-like behavior which is due to the occurrence of a triple collision, i.e., \( x_1 \) and \( x_2 \) both tend to zero simultaneously. For \( 4.7 < \theta_1 < 5.7 \) we see a very complicated dependence of \( T \) on \( \theta_1 \) which indicates the presence of chaotic scattering. Moreover, the delay time \( T \) seems to be unbounded at particular values of \( \theta_1 \). In Fig. 3 we cut the delay time at \( T = 200 \) for graphical reasons.

It is interesting to speculate how the presence of classical chaotic scattering will manifest itself on the quantum mechanical level. In the case of a "molecule" scattered off the inhomogeneous electric field of two charged wires it was demonstrated\textsuperscript{40} that classical chaotic scattering yields a quantum scattering matrix which to a first approximation can be considered a representative of Dyson's circular ensemble of random unitary symmetric matrices\textsuperscript{49–52}. Since in the stretched helium system time reversal symmetry is active, we expect to see the same behavior for the scattering matrix describing electron scattering off the one-dimensional He\textsuperscript{+} ion\textsuperscript{13}.

According to the arguments presented at the end of the previous section, we also expect to find an energy regime which exhibits Ericson fluctuations\textsuperscript{38,39}. Following first conjectures\textsuperscript{53} it was shown recently that this regime is also important in mesoscopic systems\textsuperscript{54}. In the case of the (one-dimensional) helium atom a complication arises in that close to thresholds the autoionizing resonances become sharper and sharper. Therefore, close to thresholds the imaginary parts of the scattering resonances are expected to be very small and the fluctuations, as a function of the energy, are expected to cross over into a resonance regime which we will call the "Rydberg regime". The Rydberg regime is characterized by clearly distinguishable isolated resonances. Therefore, as a function of the excitation energy we expect to observe an alternating sequence of Rydberg and Eric-
son regimes. The Ericson regime is expected to be developed between thresholds whereas the Rydberg regime will take over close to thresholds. In the cross-over region between the Ericson regime and the Rydberg regime, where the imaginary parts of the resonances slowly decrease to zero, we expect to see yet another regime which we will call the “Wigner regime”. In this regime the spacings of the S-matrix poles will be larger than the imaginary parts of the resonances which allows us to speak of “energy levels”\(^{55}\). We expect that these levels exhibit level repulsion and a Wignerian nearest neighbor statistics. In analogy to dissipative systems\(^{52,55,56}\) we expect the resonance poles in the complex plane to exhibit Ginibre statistics\(^{52,55–57}\).

According to the discussion above, quantum mechanically we expect to see the following picture as a function of the excitation energy of the one-dimensional helium atom. Below the first ionization threshold we have a regular Rydberg series of bounded states. Even the states above the first threshold, but below the second will be very regular. The same is expected up to the fifth threshold: regular sequences of “states” (auto ionizing resonances) only sporadically broken by “intruder states” from a sequence above\(^{13}\). For resonances above the fifth threshold we expect to see an increasingly well developed Wigner regime between thresholds. But since the resonances are still relatively close to the real axis\(^{13}\) we do not expect to see the Ericson regime yet. But we do expect to see a sequence of regular resonances close to thresholds. According to some very preliminary estimates\(^{13}\) the Ericson regime is expected to manifest itself around the 30’th ionization threshold from where on the energy region between ionization thresholds is expected to display the full repetitive sequence of Ericson \(\rightarrow\) Wigner \(\rightarrow\) Rydberg regimes.

In the following section we will present preliminary evidence for the existence of the Wigner regime in stretched helium.

**5. Dynamical Regimes in Stretched Helium**

In this section we will present preliminary evidence for the occurrence of a Wigner regime in doubly excited one-dimensional helium. It occurs at moderate excitation energy characterized by relatively small resonance widths. As discussed in section 2, the most elementary model for stretched helium is the “independent particle model”, which totally
neglects all correlations between the two electrons. Its energy spectrum is given by:

\[ e_{nm} = -\frac{1}{2n^2} - \frac{1}{2m^2} \]  \hspace{1cm} (5.1)

In this formula \( n \) and \( m \) are the actions of the two independent electrons, respectively. Structurally, and on the crudest level, the true spectrum of one-dimensional helium is very similar to the spectrum (5.1). Therefore, the independent particle model will serve as a point of reference in the following discussion.

For \( n \to \infty \) the energy (5.1) converges to the \( m' \)th ionization threshold \( e_m = -1/2m^2 \).

The energy levels of the one-dimensional helium atom including the electron - electron correlation energy will be denoted by \( E_{nm} \). It is useful to stratify the helium spectrum by normalizing the helium levels to the \( m' \)th ionization threshold:

\[ Y_{nm} = \frac{1}{\sqrt{-2E_{nm}}} \]  \hspace{1cm} (5.2)

For very high excitation energies we expect that the \( Y \) levels are approximately equi spaced between and not too close to ionization thresholds.

It is possible to calculate the levels of stretched helium with considerable accuracy in a complete \( L^2 \) basis\(^{13,15,26}\). For the following calculations, however, we decided to use a cheaper, only slightly less accurate, approach namely diagonalization of (2.3) in the (incomplete) basis (2.4). This approach was called the "bound space projected" approach in section 2. Comparing the results of this approach with an \( L^2 \) approach the relative accuracy is typically better than 1%.

Figs. 4-6 show energy levels obtained for the class of symmetric wave functions. The bar graph in Fig. 4 shows the levels obtained in the bound space projected approach from thresholds \( m = 1 \) to \( m = 5 \). Regular sequences of states can be identified accumulating at the respective ionization thresholds. The bar graphs in Fig. 5 and Fig. 6 show the levels corresponding to the thresholds \( m = 6, \ldots, 10 \) and \( m = 11, \ldots, 15 \), respectively. It can be seen that the level sequences between thresholds become more and more irregular. Also, the space between thresholds tends to fill in more equally as the excitation energy increases. Theoretically, the density of states diverges at all ionization thresholds. While our calculations reproduce this behavior very well for low \( m \) thresholds (see Fig. 4),
the density of states seems to thin out for higher $m$ thresholds (see Figs. 5,6). This behavior is due to numerical restrictions. Only 406 basis states were used to diagonalize (2.3). Including more basis states increases the density of states at individual ionization thresholds.

On the basis of the bar-graphs shown in Figs. 4-6, we performed a statistical analysis of nearest neighbor spacings. Fig. 4 shows the result of the spacing statistics for energy levels up to the fifth ionization threshold. Plotted is the probability distribution $P(s)$ of spacings $s$ (normalized to the mean spacing). The resulting probability distribution mainly reflects the regular sequences of states which converge to the ionization thresholds. Therefore, the probability distribution is peaked at $s = 0$ and does not correspond to any of the established universality classes$^{49-52}$.

The situation looks dramatically different for the spacings distribution of levels from the 6'th to the 10'th threshold (see Fig. 5). It can be seen that the distribution starts to move towards a Wignerian statistics.

Fig. 6 represents the level distribution for thresholds from $m = 11$ to $m = 15$. Here, the distribution is already very close to Wignerian.

As discussed in the previous section, care has to be taken as to which levels to include in the statistical analysis. Levels close to the ionization thresholds are members of the Rydberg regime and should be excluded when focusing on the Wigner regime. We have presently not established any rigorous criteria for the boundaries between the different regimes. On the other hand, our numerical calculations do not reproduce more than a few states in the Rydberg regime (see bar graphs in Figs. 5,6) anyway. Therefore, in our present calculations, none of the states shown in the bar-graphs in Figs. 4-6 was excluded and the level statistics shown in Figs. 4-6 was performed on the basis of all the states obtained from the 406 state numerical diagonalization.

6. Summary and Conclusions

Following the discussion in sections 5 and 6 and the many publications which recently appeared on the subject, there is no doubt that the dynamics of one-dimensional helium shows many fascinating features which are due to the nonlinear nature of the two-electron
problem. We presented indications that the one-dimensional helium atom is a chaotic scattering system. Moreover, we presented some preliminary results which indicate that for sufficiently high excitation energy we can expect the doubly excited one-dimensional helium atom to exhibit a repetitive sequence of three dynamical regimes: Ericson, Wigner and Rydberg. But just how strong is the evidence for the existence of these three different regimes? While the Wigner and the Rydberg regimes seem to be relatively firmly established, the evidence for the Ericson regime is at present circumstantial at best. The existence of the Wigner regime, for instance, is strongly supported by the facts presented above. Also, there seems to be no doubt about the existence of the Rydberg regime. In reference 13, using an $L^2$ approach, regular sequences of states with monotonically decreasing widths were found which converge to ionization thresholds. These states can be identified with states of the Rydberg regime. The existence of the Ericson regime, however, was extrapolated on the basis of the behavior of the widths of states associated with ionization thresholds $m \leq 10$. But since the Ericson regime was so far found in all chaotic scattering systems, we have no doubt that it will be present in the chaotic one-dimensional model of the helium atom.

We are convinced that the stretched helium model is an interesting dynamical system which shows highly complex behavior both on the classical and the quantum levels. If a one-dimensional version of the helium atom already shows such a rich dynamical behavior how much more structure is waiting to be explored in the "real" helium atom?

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References


Figure Captions

Fig. 1: Schematic sketch of the stretched helium geometry. The nucleus of charge Z is assumed to be at rest at $x = 0$ (infinite mass limit). Electron 1 is confined to the left half space, electron 2 to the right. Both electrons can interact via the repulsive electron-electron Coulomb force across the "barrier" at $x = 0$.

Fig. 2: Equipotential lines for the potential $V(x_1, x_2) = -\frac{1}{x_1} - \frac{1}{x_2} + \frac{\epsilon}{x_1 + x_2}$ for $V = -1.5$ to $V = -0.5$ in steps of $\Delta V = 0.1$ for $\epsilon = 1/2$. The potential $V$ resembles "elbow" scattering geometries recently studied in the literature.

Fig. 3: Distribution of delay times for a chaotic scattering experiment. The first electron is initially in a bounded orbit with $n_1 = 1$. The second electron is launched towards the helium nucleus with $x_2 = 10$ and $p_2 = -0.5$. Shown is the time the resulting scattering complex stays "bounded" (i.e., $x_1$ and $x_2 < 10$) as a function of the initial angle $\theta_1$ of electron 1. Irregular behavior is seen for $4.7 < \theta_1 < 5.7$ indicating the presence of chaotic scattering.

Fig. 4: Energy levels of one-dimensional helium and their statistical analysis. a) Stratified energy levels for thresholds $m = 1$ to $m = 5$. b) Nearest neighbor spacing statistics of the levels shown in (a). The histogram corresponds to the numerical data. The smooth lines are theoretical spacing distributions which correspond to the Poissonian, Wignerian, Unitary and Symplectic ensemble, respectively.

Fig. 5: Same as Fig. 4 but for energy levels corresponding to thresholds $m = 6$ to $m = 10$.

Fig. 6: Same as Fig. 4 but for energy levels corresponding to thresholds $m = 11$ to $m = 15$. 
Fig. 3